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On Gauge Invariant Wilsonian Flows^a

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Abstract

We investigate non-Abelian gauge theories within a Wilsonian Renormalisation Group approach. Our main question is: How close can one get to a gauge invariant flow, despite the fact that a Wilsonian coarse-graining seems to be incompatible with gauge invariance? We discuss the possible options in the case of *quantum* fluctuations, and argue that for *thermal* fluctuations a *fully* gauge invariant implementation can be obtained.

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1 Introduction

The Wilsonian Renormalisation Group¹ has proven itself to be a powerful tool for studying both perturbative and non-perturbative effects in quantum field theory. It has been particularly successful for scalar theories, where a number of new results have been obtained. One expects that a suitable formulation for non-Abelian gauge theories might provide new insight into non-perturbative effects in QCD as well. However, the Wilsonian approach is based on the concept of a step-by-step integrating-out of momentum degrees of freedom and one may wonder whether this concept can be adopted for gauge theories. Indeed, one of the main obstacles of the Wilsonian approach to gauge theories was precisely the question as to how gauge invariance can be controlled.

In this contribution we analyse this question in some detail. In particular, we want to understand how close one might come to a gauge invariant implementation of a Wilsonian coarse-graining in the usual sense and which net gains are related to such procedures.

This contribution is organised as follows: The first parts consider generic features of Wilsonian flows for gauge theories. We discuss the modified Ward Identities, in particular the resulting constraints for a construction of gauge invariant flows, and we detail the link between the standard approach and the background field methods. Finally we comment on the key features of the different gauge invariant approaches. In the remaining part, we investigate the application to thermal field theory, and argue why a gauge invariant implementation is possible for all scales in the case of thermal fluctuations.

2 The flow equation

We investigate these questions in a path integral approach based on ideas of Polchinski². In this approach a momentum cut-off is achieved by adding a cut-off term $\Delta_k S$ to the action which is quadratic in the field. This results in an effective action Γ_k where momenta larger than k have been integrated-out. The change of Γ_k under an infinitesimal variation of the scale k is described by a flow equation which can be used to successively integrate-out the momenta smaller than the cut-off scale k . Thus given an effective action Γ_{k_0} at an initial scale k_0 the flow equation provides us with a recipe how to calculate the full effective action Γ .

The introduction of $\Delta_k S$ seems to break gauge invariance. However, Γ_k satisfies a modified Ward identity (mWI). This mWI commutes with the flow and approaches the usual Ward identity (WI) as $k \rightarrow 0$. Consequently the full effective action Γ satisfies the usual Ward identity. In other words, gauge invariance of the full theory is preserved if the effective action Γ_{k_0} satisfies

the mWI at the initial scale k_0 . Let us see how close to a gauge invariant description one might get even during the flow, that is for all k .

2.1 Derivation of the flow equation

To that end let us outline the derivation of the flow equation. In the following we employ the superfield formalism as introduced in³. A more explicit version of the following arguments may be found in^{4,5,6,7}. The starting point is the infrared (IR) regularised Schwinger functional

$$\exp W_k[J] = \frac{1}{\mathcal{N}_k} \int d\phi \exp\{-S_k[\phi] + \text{Tr } \phi^* J\}, \quad (1)$$

where the trace Tr denotes a sum over momenta, indices and the different fields ϕ , including a minus sign for fermionic degrees of freedom. $(\phi_i) = (\phi_1, \dots, \phi_s)$ is a short-hand notation for all fields and ϕ^* denotes its dual. The term $S_k[\phi]$ contains the (gauge-fixed) classical action and a quadratic cut-off term $\Delta_k S[\phi]$ given by (e.g.⁴ and references therein):

$$\Delta_k S[\phi] = \frac{1}{2} \text{Tr} \left\{ \phi^* R_k^\phi [P_\phi] \phi \right\}, \quad (2)$$

where P_ϕ^{-1} is proportional to the bare propagator of ϕ . The k -dependent constant \mathcal{N}_k has been introduced to guarantee an appropriate normalisation of W_k . The flow of (1) related to an infinitesimal change of k is given by ($t = \ln k$)

$$\partial_t \exp W_k[J] = -\frac{1}{\mathcal{N}_k} \int d\phi \left(\frac{1}{2} \text{Tr } \phi^* R_k^\phi \phi + \frac{\partial_t \mathcal{N}_k}{\mathcal{N}_k} \right) e^{-S_k[\phi] + \text{Tr } \phi^* J}. \quad (3)$$

Eventually we are interested in the Legendre transform of W_k , the effective action Γ_k :

$$\Gamma_k = \text{Tr } \phi^* J - W_k[J] - \Delta_k S[\phi]. \quad (4)$$

For the effective action Γ_k the flow equation (3) turns into

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left\{ G_k^{\phi^* \phi} [\phi] \partial_t R_k^\phi [P_\phi] \right\} + \partial_t \ln \mathcal{N}_k \quad (5)$$

with

$$G_k^{\phi_i^* \phi_j} [\phi] = \left(\frac{\delta^2 \Gamma_k[\phi]}{\delta \phi_i^* \delta \phi_j} + R_{k,ij}^\phi [P_\phi] \right)^{-1}. \quad (6)$$

The flow equation (5) and an initial effective action Γ_{k_0} may be read as a proper definition of the path integral. In order to have a well-defined flow equation we use operators R_k^ϕ with the following properties ($x = P_\phi k^{2d_\phi-d}$):

$$k^{2d_\phi-d} R_k^\phi[P_\phi] \xrightarrow{x \rightarrow 0} \frac{x^n}{|x|}, \quad n \leq 1 \quad (7)$$

$$x^{d/(2d_\phi-d)} P_\phi^{-1} R_k^\phi[P_\phi] \xrightarrow{x \rightarrow \infty} 0, \quad (8)$$

where d_ϕ are the dimensions of the fields ϕ . A very interesting subset of operators R_k with the properties (7),(8) is the set of operators which satisfy $n = 1$ in (7) and decay exponentially for $x \rightarrow \infty$. These lead to very good convergency properties if invoked for numerical calculations. A cut-off term (2) with R_k satisfying (7),(8) effectively suppresses modes with momenta $p^2 \ll k^2$ in the generating functional. For modes with large momenta $p^2 \gg k^2$ the cut-off term vanishes and in this regime the theory remains unchanged. In the limit $k \rightarrow 0$ we approach the full generating functional Γ since the cut-off term is removed. In the limit $k \rightarrow \infty$ all momenta are suppressed and –with a suitably chosen \mathcal{N}_k – the effective action approaches the (gauge fixed) classical action $S_{\text{cl}} + S_{\text{gf}}$. Hence Γ_k interpolates between the classical action and the full effective action:

$$S_{\text{cl}} + S_{\text{gf}} \xleftarrow{k \rightarrow \infty} \Gamma_k \xrightarrow{k \rightarrow 0} \Gamma. \quad (9)$$

Note that $\partial_t R_k^\phi$ serves as a smeared-out δ -function in momentum space peaked at about $p^2 \approx k^2$. In this sense, a Wilsonian flow is local, as only a small window of momentum modes around k will contribute to the flow. By varying the scale k towards smaller k according to (5) one successively integrates-out momentum degrees of freedom.

3 Control of gauge invariance

In this section we show how gauge invariance is –in general– controlled by modified Ward Identities (mWI). For perturbative truncations to the effective action Γ_k , the mWI can be maintained employing the quantum action principle. The more important point is that the mWI can also be satisfied within non-perturbative truncations.

3.1 Modified Ward Identities

When applied to gauge theories the cut-off term (2) generates additional terms in the Ward identity thus leading to a modified Ward Identity. For non-

Abelian gauge theories formulated in general linear gauges $L_\mu A_\mu$ and gauge fixing parameter ξ , one can derive the following mWI^{4,8}

$$\begin{aligned}\mathcal{W}_k[\phi] &= \delta_\alpha \Gamma_k[\phi] - \text{Tr} (L_\mu D_\mu \alpha) \frac{1}{\xi} L_\nu A_\nu + \frac{g}{2} \text{Tr} \left[\alpha, \left(R_k^\phi + L^* \frac{1}{\xi} L \right) \right] G_k^{\phi^* \phi} \\ &= 0,\end{aligned}\tag{10}$$

where we used the abbreviation δ_α for the generator of infinitesimal gauge transformation on ϕ , e.g. $\delta_\alpha A = D(A)\alpha$ for the gauge field and $\delta_\alpha \psi = \alpha\psi$, $\delta_\alpha \varphi = \alpha\varphi$ for fermions and scalars respectively. Note also that α has to be taken in the representation of the corresponding field ϕ_i . For the sake of brevity we dropped any reference to possible ghost fields which have to be added in general; however they have to be treated in a similar manner.

The cut-off dependent terms in (10) vanish for $k \rightarrow 0$. The compatibility of the mWI (10) with the flow equation is given by^{4,8}

$$\partial_t \mathcal{W}_k[\phi] = -\frac{1}{2} \text{Tr} \left(G_k^{\phi_i^* \phi_j} \partial_t R_k^{\phi_j} G_k^{\phi_j^* \phi_l} \frac{\delta}{\delta \phi_l^*} \frac{\delta}{\delta \phi_i} \right) \mathcal{W}_k[\phi]. \tag{11}$$

Eq. (11) ensures that if the initial effective action satisfies the mWI (10) than the usual Ward identity is satisfied for $k = 0$. It is now possible to employ the quantum action principle. One can solve the mWI (10) order by order in the coupling. This solution will remain a solution for any k due to (11). In other words, an effective action Γ_{k_0} that solves the mWI up to order n in the coupling stays a solution up to order n for any k_0 , in particular for $k = 0$. This is how perturbation theory is included as one possible truncation scheme (see also⁹).

In general one may use another expansion parameter –instead of the coupling– which is small in the regime of interest, e.g. p^2/Λ_{QCD}^2 in the deep infrared regime of QCD.

3.2 Numerical implementation

The flow equation (5), the mWI (10) and a suitably truncated (initial) effective action are the starting points for numerical applications^{3,10}. The first step is to introduce a parametrisation of the effective action in terms of some couplings γ . In an expansion in the powers of the fields, these couplings are just the momentum-dependent vertex functions. The key point is that the mWI introduces relations between the different couplings γ . Thus only a subset of couplings $\{\gamma_{\text{ind}}\}$ can be independently fixed, whereas the other couplings γ_{dep}

can be derived with the mWI and the set $\{\gamma_{\text{ind}}\}$. It is worth mentioning that the splitting $\{\gamma\} = \{\gamma_{\text{ind}}, \gamma_{\text{dep}}\}$ is not unique.

Now we chose a truncation of the flow equation such that only the couplings γ related to operators important for the problem under investigation are included. For QCD, these typically include the gauge coupling, a gluonic mass term, and the 3- and 4- point (and higher) vertices. In general, only a (finite) subset of $\{\gamma_{\text{dep}}(k)\}$ is taken into account. The flow equation will then be integrated as follows: After integration of an infinitesimal momentum shell between $k + \Delta k$ and k we obtain couplings $\gamma_{\text{ind}}(k)$. With the mWI one derives the (finite) subset of $\{\gamma_{\text{dep}}(k)\}$ which together with the $\gamma_{\text{ind}}(k)$ serve as the input for the next successive integration step. By employing this procedure for the integration of the (truncated) flow equation from the initial scale k_0 to a scale k one obtains a set of $\{\gamma(k)\} = \{\gamma_{\text{ind}}(k), \gamma_{\text{dep}}(k)\}$ for any scale k . These $\gamma(k)$ parametrise an effective action Γ_k which by construction does satisfy the mWI at any scale k , in particular for $k = 0$ (see section 3.1). Thus the full effective action Γ_{trunc} calculated with the truncated flow equation satisfies the usual Ward identity. The truncation does not imply a breaking of gauge invariance but rather a neglecting of the back-reaction of the truncated couplings on the flow of the system.

For a validity check of the truncation we have to employ the fact that the system is overdetermined. The set $\{\gamma_{\text{dep}}\}$ may be also directly calculated with the flow equation (5) itself. Only for the full system both equations (flow equation and mWI) are compatible, as shown in section 3.1, Eq. (11). Thus as long as the results for the γ_{dep} , which are obtained by either using (5) or using (10), do not deviate from each other, the truncation remains valid. The validity bound of a truncation is reached when these independently determined results for γ_{dep} no longer match. Typically, this defines a final cut-off scale $k_{\text{fin}} \ll k_0$. Such a check has been done with the gluonic mass³. Even though this was only a partial consistency check –and thus not entirely satisfactory– it essentially gives the flavour of what has to be done in practice: For non-perturbative truncations the mWI is employed both for the consistency check *and* as a tool in order to calculate the value of the γ_{dep} . As mentioned in the last section, for a fully controlled calculation one additionally has to find a suitable expansion parameter which can be employed for general validity checks of the truncation.

4 Analytic methods

In this section we detail how the flow equation can be employed to derive non-trivial analytic results. We make use of the background field approach, and, subsequently, of general axial gauges.

4.1 Background field formalism

Let us first show how the usual Ward identity is obtained by using a background field⁸. The following derivation is strictly valid only for momentum independent gauges; however a minor modification of it also applies to general linear gauges⁵. The background field dependence in this approach is given by an equation quite similar to the flow equation:

$$\frac{\delta}{\delta \bar{A}} \partial_t \Gamma_k = \frac{1}{2} \partial_t \text{Tr} \left\{ G_k^{\phi^* \phi}[\phi, \bar{A}] \frac{\delta}{\delta \bar{A}} R_k^\phi \right\}. \quad (12)$$

This translates into the following equation for an infinitesimal gauge transformation of \bar{A} applied on the effective action:

$$\bar{\delta}_\alpha \Gamma_k[\phi, \bar{A}] = \frac{g}{2} \text{Tr} \left\{ [\alpha, R_k^\phi] G_k^{\phi^* \phi} \right\}, \quad (13)$$

where $\bar{\delta}_\alpha$ is defined by its action on the fields, e.g. $\bar{\delta}_\alpha = D(\bar{A})\alpha$, $\bar{\delta}_\alpha \phi = 0$. It follows from (10), (13) that $\Gamma_k[\phi, \bar{A}] - S_{\text{gf}}[A]$ is invariant under the transformation $(\delta_\alpha + \bar{\delta}_\alpha)$. As a consequence $\hat{\Gamma}_k[\phi] := \Gamma_k[\phi, \bar{A} = A]$ satisfies the usual WI without the cut-off dependent terms:

$$\delta_\alpha \hat{\Gamma}_k[\phi] = \text{Tr} [L_\mu D_\mu(A) \alpha] \frac{1}{\xi} L_\nu A_\nu. \quad (14)$$

where the gauge field derivative involved in (14) hits both the gauge field A and the auxiliary field $\bar{A} = A$. Note however that the propagator G_k^{AA} is still the one derived from $\frac{\delta^2}{\delta \bar{A}^2} \Gamma_k[\phi, \bar{A}]$ at $\bar{A} = A$. The flow equation for $\hat{\Gamma}_k$ requires the knowledge of G_k^{AA} , thus slightly spoiling the advantage of dealing with an effective action which satisfies the usual WI even for $k \neq 0$.

4.2 General axial gauges

General axial gauges have recently been studied within this approach^{4,5,8}. It has been established that the spurious singularities of perturbation theory are absent⁴. This implies that the theory is well-defined without any further regularisation in contrast to standard perturbation theory. Other advantages are the absence of Gribov copies and the decoupling of the ghost sector. Finally, the gauge fixing parameter ξ has a non-perturbative fixed point at $\xi = 0$, which makes this formulation very attractive for both analytical and numerical computations.

In the following we refer to analytic calculations done in a general axial gauge and using the background field approach as briefly discussed in section 4.1. As a consequence of (14) we have gained gauge invariance –apart from the explicit breaking from the gauge fixing– even for $k \neq 0$. This simplifies the expansion of the effective action. The problem is now to distinguish between the gauge field A and the field $\bar{A} = A$ which only serves as an auxiliary variable. This is necessary since the flow equation still requires the knowledge of G_k^{AA} as mentioned above, and which can be obtained at least in principle from (12). Additionally the \bar{A} -dependence of the effective action should be dropped completely since in the present approach it is *only* an auxiliary variable and for $k = 0$ there is no \bar{A} -dependence at all.

Let us illustrate the importance of the latter point with the following example of the 1-loop β -function: It can be shown with (12) (apart from higher order terms in \bar{A}) that the flow equation on 1-loop level leads to a contribution proportional to $(n-1) \int F^2(\bar{A})$ to Γ_k ⁵, where n has been defined in (7). This vanishes for operators R_k with masslike IR limit ($n = 1$) even though it is in general non-zero beyond 1-loop level. For these R_k the 1-loop β -function is straightforwardly calculated. However for R_k satisfying (7) with $n \neq 1$ this term is non-vanishing and has to be subtracted from $\bar{\Gamma}_k$ as calculated with the flow equation. In principle this can be done by using an appropriately defined \mathcal{N}_k . Subtracting the contribution proportional to $(n-1) \int F^2(\bar{A})$ from the flow equation leads to the correct 1-loop β -function and other 1-loop quantities.

With (5), (10), (12) and (13) we can investigate the effective action analytically. It is worth noting that the flow equation is a ‘1-loop’ equation, even though the loops depend on the full field dependent propagator. Thus heat kernel methods can be employed, although *not* as a regularisation method, since everything is finite from the onset. Restricted to the perturbative regime, these calculations^{4,5,8} yield not only the perturbative β -function for a non-Abelian gauge theory coupled to fermions for arbitrary gauge fixing parameter ξ , but results beyond the 1-loop level as well.

5 Gauge invariant flows

Let us now come back to the question about gauge invariant flows. The standard formulation, based on the flow and mWI necessitates the introduction of some gauge non-invariant operators for non-vanishing k , like a gluonic mass term, in order to assure gauge invariant physical Green function. It would be interesting to see under which circumstances gauge non-invariant operators can be avoided.

The important observation to that end is, that usual gauge invariance for

any scale k can only be achieved by relaxing at least one of the key assumptions leading to the flow equation itself. There are essentially three options available. One either relaxes the constraints regarding the regulator function R_k ¹¹, or starts with a completely different mechanism for introducing the regularisation in the first place¹², or introduces auxiliary fields^{5,6,8,13}. We shall now discuss all these options in more detail.

5.1 Momentum independent regulator

Let us comment on the first option, that is to change the requirements regarding R_k . It is straightforward to observe that a necessary and sufficient condition for usual gauge invariance even during the flow is just the vanishing of the commutator $[R_k^\phi + L^* \frac{1}{\xi} L, G_k^{\phi\phi^*}[\phi]] = 0$ (see (10)). The only solution to this constraint (apart from the necessity of a momentum independent gauge fixing) is $R_k^\phi = \propto k^{d-2d_\phi}$ thus introducing mass terms proportional to the cut-off scale k . Even though this choice satisfies (7) (with $n = 1$), the second condition (8), which guarantees that the ultraviolet (UV) behaviour of the theory is unaltered in the presence of the cut-off term, is no longer satisfied.

The kernel of the trace in the flow equation (5) is no longer peaked at momenta about k , if (8) is violated. Even more so, (5) is not well-defined as it stands and needs some additional UV renormalisation. To be consistent, this has to be done on the level of the effective action rather than on the level of the flow equation. Otherwise the connection between the flow equation and the original –even though only formal– path integral becomes unclear. Furthermore, since this additional UV renormalisation has to be k -dependent, one may ultimately lose the 1-loop structure of the flow equation. This depends on how the actual renormalisation is done.

Moreover the interpretation of the flow equation is now completely different from the original Wilsonian idea. The flow equation no longer describes a successive integrating-out of momentum modes, but rather a flow in the space of massive theories. Even though the suppression of low momentum modes still works at every step of the flow, all parameters of the theories change for *all* momenta larger than k . This can be considered as a loss of locality, in the sense mentioned earlier. It has also to be pointed out that the result is *not* what is usually denoted by a massive gauge theory. The difference stems from the fact that the cut-off term –in the Wilsonian approach– is introduced *after* the gauge fixing has been done. In the case of a massive gauge theory the Fadeev-Popov mechanism is applied to the path integral, where the action already includes the mass term. The difference between these two approaches are those terms stemming from $\int dg \exp -k^2 \text{Tr}(A^g)^2$, where $g(x)$ is a space-time dependent

gauge group element. These are exactly the terms which usually are made responsible for the breakdown of renormalisability in massive gauge theories. A simple way to see this is as follows: Introduce $\chi, g = e^\chi$ as a new field and do the usual power counting with respect to the fields (A, χ) . Dropping these terms changes the content of the theory, and although it looks superficially like a massive gauge theory, it is *not*.

Apart from these conceptual problems, it appears that a numerical implementation is essentially out of reach. The momentum integrals involved would receive contributions from *all* momenta larger than k instead of only being peaked within a small momentum shell at about $p^2 \sim k^2$. Note that the numerical applicability of the Wilsonian flow equation may be seen as one of its most attractive features, and losing it is a big loss for gaining formal gauge invariance during the flow.

It should be mentioned that a mass-like regulator remains an interesting option for a first approximative computation, consistency checks or conceptual issues, as it typically simplifies analytic calculations tremendously. For more involved and non-trivial truncations in general, one has to use more elaborate regulators, though.

5.2 Gauge invariant variables

A more attractive possibility is the proposal to change the starting point of the derivation, but to stick to the 1-loop nature of the resulting flow equation. This can only be done by mapping the degrees of freedom from the original fields to another set in a non-linear way, e.g. to a representation in terms of Wilson loops¹². It is worth noting that this particular procedure¹² requires an additional UV renormalisation which has been done explicitly by introducing Pauli-Villars fields. It also requires the introduction of a second gauge field. All these auxiliary fields only decouple in the limit $k \rightarrow 0$. Moreover, at least one of these fields needs a mass-like cut-off which again spoils the locality of the flow as defined earlier.

Thus the same pitfalls concerning the applicability as in the other approach do finally apply here as well. Nevertheless –even though for finite k the number of field degrees of freedom is considerably enlarged– it seems to be possible to track down the original degrees of freedom at any cut-off scale k .

5.3 Background fields

Finally, we will discuss the third option –which has actually been used first⁶ – and which consists in introducing auxiliary fields^{5,6,8,13}. This is nothing but the adaption of the well-known background field formalism to a theory where

a cut-off term is present. The key point here is to introduce the covariant derivative with respect to the background field \bar{A} wherever a plain derivative was used in R_k . As a consequence, the effective action now satisfies the usual Ward identity if *all* fields are gauge transformed (The particular gauge transformations needed differ slightly in the various approaches).

Even though the flow equation is still peaked at (covariant) momenta about k it is obvious that the dependence of the effective action on the background field is non-trivial. One has to employ an additional equation to track down this dependence, which adds up with the flow equation to a bigger set of non-trivial equations. This is how in this approach the necessity of applying a non-linear transformation in field space shows up. At least the usefulness for numerical implementations seems to be questionable.

In summary one may conclude that the price for gauge invariance during the flow is rather high for all these approaches, in particular in comparison to the standard approach using the mWI. When it comes to numerical applications the latter approach seems clearly preferable.

6 Quantum fields at non-vanishing temperature

We will now discuss an application of the Wilsonian Renormalisation Group to thermal field theory. The aim is to show that, in contrast to the previous sections which dealt with *quantum* fluctuations, for *thermal* fluctuations a fully gauge invariant flow can be constructed.

6.1 How does temperature enter a quantum field theory

We shall start with some general remarks on quantum field theories coupled to a heat bath. The temperature is introduced via the compactification of the (imaginary) time direction. This implies that the fields ϕ live in the space $[-1/T, 1/T] \times \mathbb{R}^3$ rather than \mathbb{R}^4 . In addition, this results in the imposition of periodic (anti-periodic) boundary conditions for the bosonic (fermionic) degrees of freedom.

Having said that, it follows that the modes with momenta much larger than the temperature will not be aware of the altered boundary condition, i.e. they will behave like modes living again in \mathbb{R}^4 , that is like zero temperature modes. Stated differently, the UV physics is unaltered, and in particular, no further UV divergences than those encountered for the bare action at vanishing temperature will be observed. The low momentum or soft modes, however, do feel the new boundary condition. It is precisely for this reason that temperature has to be considered as an IR phenomenon.

6.2 How to detect thermal corrections

In order to identify the effects imposed by temperature, we have to introduce a reference point. Typically, a physical observable computed for some fixed temperature has to be compared to that very same observable computed at some other temperature, say $T = 0$. Only their difference can be given the desired physical meaning. Thus, the effects of temperature are detected as a difference between observables measured for different boundary (temperature) conditions. This amounts to the statement that the natural object to study is

$$\Gamma_T[\phi] - \Gamma_{T=0}[\phi] , \quad (15)$$

where Γ_T denotes the effective action of a given quantum field theory at some fixed temperature T . Note that the proper definition of (15) requires an appropriate definition of the space of fields ϕ . In other words, the finite temperature fields ϕ for which (15) makes sense have to be properly embedded in the zero temperature space-time¹⁷. Also physical observables derived from Γ_T may be of interest. A very important example is the thermal pressure

$$P[T] - P[0] . \quad (16)$$

It is related to the minimum of the effective potential, which itself is the leading order term within a derivative expansion of Γ_T . For theories with their potential minimum at vanishing field, the pressure corresponds precisely to the field independent part of (15).

6.3 How to compute thermal corrections

The more difficult question is now how (15) actually can be computed. Consider for example the thermal pressure: A perturbative computation of the pressure for a scalar field theory faces strong IR divergences. A systematic resummation has to be performed in order to obtain a finite result. Even worse is the case of QCD. Its perturbative computation is severely limited due to the non-perturbative magnetic sector of QCD. These are generic examples for the problems of perturbative loop expansions at finite temperature, which do typically encounter serious IR problems at some loop order. In addition, it also has been observed that the convergence of the series for the thermal pressure is rather poor even in the domain, where perturbation theory should be applicable.

7 Wilsonian flow for the thermal fluctuations

We want to construct, based on Wilsonian ideas, a non-perturbative resummation procedure to compute (15). Applications of Wilsonian flows to thermal field theories are not new¹⁴. The usual approach is as follows: One constructs a flow equation for $\Gamma_{k,T}$ which has to be solved (in some approximation) to yield Γ_T . Within imaginary time, the flow equation is obtained from (5), replacing the trace by

$$\text{Tr} = \sum_{\phi} \int \frac{d^3 p}{(2\pi)^3} T \sum_n \quad (17)$$

i.e. a sum over all momenta, Matsubara modes, and all (bosonic) fields and their indices. The substitution $p_0 \rightarrow 2n\pi T$ for bosons is also understood. Then, one uses either the same flow but with $T = 0$, or some alternative method like resummed perturbation theory, to compute $\Gamma_{T=0}$. Combining these results, one finally obtains (15).

7.1 Flow equation for the thermal contribution

In contrast to the standard approach, we now aim at a flow equation not for $\Gamma_{k,T}$, but directly for

$$\Delta\Gamma_{k,T}[\phi] = \Gamma_{k,T}[\phi] - \Gamma_{k,0}[\phi] . \quad (18)$$

The difference (18) effectively projects-out the thermal fluctuations. In the IR limit $k \rightarrow 0$, (18) reduces to (15), which is precisely what we are looking for. Note that in order to evaluate (18), we have fixed the temperature T for all fields occurring in (18). Given the flow (5), it is straightforward to write down a corresponding flow for (18), which reads for bosonic fields

$$\begin{aligned} \partial_t \Delta\Gamma_{k,T}[\phi] = \partial_t \Delta \ln \mathcal{N}_{k,T} + \frac{1}{2} \sum_{\phi} \int \frac{d^3 p}{(2\pi)^3} \left\{ T \sum_n G_{k,T}^{\phi\phi*}[\phi] \partial_t R_k^{\phi} \right. \\ \left. - \int \frac{dp_0}{2\pi} G_{k,0}^{\phi\phi*}[\phi] \partial_t R_k^{\phi} \right\} . \quad (19) \end{aligned}$$

It is worth pointing out the rôle of the normalisation constant $\mathcal{N}_{k,T}$ for (18). In the standard approach (5), it can simply be neglected because it corresponds to a shift of the zero point energy. However we have already encountered an example at zero temperature where it is essential to take into account the flow of the normalisation \mathcal{N}_k (see section 4.1). There it was related to the particular choice of the regulator R_k . At finite temperature, however, $\Delta\Gamma_{k,T}$ at vanishing

field measures a zero point energy *difference*, thus a physical observable. This implies that the normalisation properly has to be taken into account.

However, up to now little has been gained while studying (19) instead of (5). For a general regulator R_k , the same qualitative problems regarding gauge invariance for arbitrary scale k are encountered as at vanishing temperature. We shall now argue, that studying (19) allows us to relax slightly one of the condition on the regulator function.

7.2 Momentum independent regulator

It was discussed earlier that a mass-like (i.e. momentum independent) regulator, employed for (5), is not viable, the reason being the uncontrolled large momentum contribution to the flow. In (19), the situation is now different. Consider a mass-like regulator $R_k^\phi = k^2$ (and a momentum independent gauge fixing), for which the flow is given by¹⁷

$$\frac{\partial \Delta \Gamma_{k,T}[\phi]}{\partial k^2} = \frac{1}{2} \sum_{\phi} \int \frac{d^3 p}{(2\pi)^3} \left\{ T \sum_n G_{k,T}^{\phi\phi*}[\phi] - \int \frac{dp_0}{2\pi} G_{k,0}^{\phi\phi*}[\phi] \right\} + \partial_{k^2} \Delta \ln \mathcal{N}_{k,T} . \quad (20)$$

First of all, the flow (20) is well defined in the IR limit. This is so, because using a mass-like regulator indeed cures the IR behaviour for the individual flows in (20), and computing the difference does not change this property. The important observation is that the flow is also well-defined in the UV limit. This is so because the remaining UV divergences, which are not eliminated by the mass-term regulator, are eliminated through the subtraction of the $T = 0$ counter part. Note that this implies that also for the initial scale k_0 the difference $\Delta_{k,T} \Gamma$ has to be local in the sense as defined in section 2.1. Thus, for large momenta, the r.h.s. of (20) does not feel the presence of a thermal bath. Stated differently, in the standard approach, the decay of $\partial_t R_k$ for large momenta ensures the UV finiteness of the flow equation. For a mass-like regulator $\partial_t R_k$ is proportional to R_k and in particular not suppressed for large momenta. However, the suppression now comes from the cancellation at high momentum between the two propagators. This establishes that (20) is a well-defined *Wilsonian* flow for thermal fluctuations.

It is worth mentioning that the cancellation of UV divergences is quite similar to the one employed in the BPHZ-procedure. The subtraction of possibly divergent terms takes place on the level the integrand rather than on the level of the regularised full expressions. This is how the explicit introduction of an UV renormalisation scale is avoided.

7.3 Gauge invariance

Let us now see what happens in the case of pure gauge theories. We consider a $SU(N)$ gauge theory in an axial gauge $(U_\mu A_\mu)$. As mentioned before it is also necessary to take a momentum independent gauge fixing parameter ξ , which is known to have at $\xi = 0$ a non-perturbative fixed point⁴. The mWI for $\Gamma_{k,T}[A]$ is derived from (10) for a general regulator as

$$\delta_\alpha \Gamma_{k,T}[A] = \int \frac{d^3 p}{(2\pi)^3} T \sum_n \left(\frac{1}{U^2 \xi} U_\mu \partial_\mu \alpha U_\nu A_\nu - \frac{g}{2} [\alpha, R_k^A] G_{k,T}^{AA}[A] \right). \quad (21)$$

The mWI for (19) follows straightforwardly from (21) as

$$\delta_\alpha \Delta \Gamma_{k,T}[A] = \frac{g}{2} \int \frac{d^3 p}{(2\pi)^3} \left(\int \frac{dp_0}{2\pi} [\alpha, R_k^A] G_{k,0}^{AA}[A] - T \sum_n [\alpha, R_k^A] G_{k,T}^{AA}[A] \right). \quad (22)$$

Note, that the term in (21) generated from the gauge fixing has canceled, and the only remaining breaking of gauge symmetry comes from the regulator function. Within the derivation of (22) it is important to use the correct boundary conditions for α (for details see¹⁷). However, with the mass-like regulator as used in (20), the mWI reads

$$\delta_\alpha \Delta \Gamma_{k,T}[A] = 0, \quad (23)$$

because the commutator $[\alpha, R_k^A]$ vanishes for momentum independent regulator. Note, that the WI controls only the gauge invariance for the field dependent part of the effective action. As mentioned before the normalisation has to be properly chosen which is a well-known problem in gauge theories at finite temperature¹⁸. As a result $\Delta \ln \mathcal{N}_k[\xi]$ is ξ -dependent. It turns out that it is precisely this ξ -dependence which guarantees that the field independent part of (19) is ξ -independent and thus gauge independent. This of course ensures gauge independence for the field independent part $\Delta \Gamma_{k,T}[0]$.

We have established the following important result: The flow equation (20) is a) a well-defined flow in the Wilsonian sense and b) explicitly maintains gauge invariance for arbitrary scale k .

7.4 Remarks

We will finish this section with some remarks. Note first that studying (15) makes an additional implicit assumption, which is that the physical degrees of freedom describing the zero temperature theory are assumed to remain reasonable degrees of freedom at temperature T . Of course, this is the prerequisite

to give the difference (18) a meaning. This does not exclude the introduction of composite operators at some intermediate scale as well. In the present formulation, this has to be done both for the $T = 0$ and the $T \neq 0$ theory at some fixed scale k_b .

A second comment concerns the initial conditions. In contrast to the usual approach, the flow for (18) can no longer be a functional differential equation for $\Delta\Gamma_{k,T}$ only. This is evident from the definition of $\Delta\Gamma_{k,T}$ in the first place. The r.h.s. will necessarily imply information about the underlying theory, for which the thermal fluctuations are computed. This is reflected by the fact that the flow equation will depend as well on $\Gamma_{k,0}$. Alternatively, this can be deduced from the initial condition. In the UV limit, we have

$$\lim_{k \rightarrow \infty} \Delta\Gamma_{k,T}[\phi] = 0 \quad (24)$$

while the IR limit reads

$$\lim_{k \rightarrow 0} \Delta\Gamma_{k,T}[\phi] = \Gamma_T - \Gamma_{T=0} . \quad (25)$$

Eq. (24) implies, that, in order to specify the theory, one has to furnish some information regarding the theory at vanishing temperature, that is on $\Gamma_{T=0}$. This is very similar to another, but qualitatively different proposal within real-time thermal field theory by d’Attanasio and Pietroni¹⁵. There, the starting point needs the renormalised $T = 0$ theory as an input as well.

Finally, it is worth noting the similarity between our flow (20), and a recent proposal to compute the thermal pressure for scalar theories¹⁶. There, the authors presented an IR finite resummation formula for the thermal pressure, which is expressed in terms of a mass integral. If we restrict ourselves to scalar fields only, and to the leading order in a derivative expansion, then our flow (20) corresponds precisely to their proposal. In this sense, our flow is the generalisation to gauge theories and the entire effective action.

8 Discussion and outlook

For the integrating-out of *quantum* fluctuations, the present status is as follows: There are several possibilities to rescue usual gauge invariance even during the flow, but all of them have to face serious problems – at least when (numerical) applicability is demanded. The reason is that the only way to gain gauge invariance within a Wilsonian formulation is to employ non-linear transformations in field space or even to lose locality –as defined in section 5– of the flow equation. The resulting formulations are quite difficult to handle if it comes to non-trivial applications. The technical price for gauge invariance along the

flow is quite big as opposed to the approach using the mWI. The handling of the latter, in particular, does not need other techniques than those already used for the flow equation itself.

It is therefore feasible to employ –for more involved questions– the regulators R_k satisfying the conditions (7),(8) along with the mWI to guarantee gauge invariance for physical Green functions. After all, the flow equation should rather be seen as a well-defined procedure as to how to calculate the full quantum effective action. This does not necessitate usual gauge invariance *during* the flow but the *control* of gauge invariance for the physical Green functions, that is, at $k = 0$.

All the approaches mentioned in section 5 certainly have their merits for particular applications and conceptional issues. Further investigations in these directions are needed in order to clarify some outstanding conceptional questions, e.g. concerning perturbative β -functions beyond 1-loop⁵, the full-fledged inclusion of topological non-trivial configurations¹³ and anomalies¹⁹.

For the integrating-out of *thermal* fluctuations, we have been able to construct a gauge invariant renormalisation group equation, based on a mass-like regulator term in combination with an axial gauge fixing. The proposal is very similar to –and can be seen as a generalisation to gauge theories of– an approach advocated earlier for scalar theories¹⁶. This proposal differs qualitatively from another gauge-invariant thermal renormalisation group introduced earlier¹⁵.

The mass-like regulator typically allows analytic computations, as has been done for the thermal pressure for scalar theories¹⁶. Further applications in thermal field theory should include the computation of the thermal pressure for a gas of bosons and gluons. In a first approximation, it suffices to insert the classical action S at $T = 0$ into the flow equation instead of $\Gamma_{k,T=0}$. The higher order corrections are then obtained through the back coupling between an improved action at vanishing temperature, and the flow equation.

More generally, it would be very interesting to see whether the Hard Thermal Loop effective action could be used as a starting point, rather than the $T = 0$ one. This might be a good starting point to go beyond the HTL approximation. We hope to report on these matters in the future.

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